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Determination of the $D^{1/2}$-Norm of the SOR Iterative Matrix for the Unsymmetric Case

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Abstract. This paper is concerned with the determination of the Jordan canonical form and $D^{1/2}$-norm of the SOR iterative matrix derived from the coefficient matrix A having the form

$$A = \begin{pmatrix} D_1 & -H \\ H^T & D_2 \end{pmatrix}$$

with D_1 and D_2 symmetric and positive definite. The theoretical results show that the Jordan form is not diagonal, but has only q principal vectors of grade 2 and that the $D^{1/2}$-norm of $\mathscr{L}_{\omega_b}$ (ω_b, the optimum parameter) is less than unity if and only if $\bar{\mu} = \rho(B)$, the spectral radius of the associated Jacobi iterative matrix, is less than unity. Here q is the multiplicity of the eigenvalue $i\bar{\mu}$ of B.

1. Introduction. For the iterative solution of the linear system of equations,

$$Ax = b, \tag{1.1}$$

the Jacobi and Gauss-Seidel methods are well known. They are very simple from a computational point of view since only matrix-vector multiplications and linear combinations of vectors are needed. This is also valid for the modification called "Successive Overrelaxation" or "SOR" method, where a relaxation factor is introduced for accelerating the convergence. Let

$$A = D - A_L - A_U, \tag{1.2}$$

where D is the block diagonal part of A, $-A_L$ and $-A_U$ are the remaining strictly lower and upper triangular parts of A; then, if D is nonsingular, the SOR method is given by

$$x_{k+1} = \mathscr{L}_\omega x_k + \omega(I - \omega L)^{-1} D^{-1} b, \qquad k \geq 0. \tag{1.3}$$

Here, x_0 is an initial vector, $\mathscr{L}_\omega$ the iterative matrix given by

$$\mathscr{L}_\omega = (I - \omega L)^{-1}[(1-\omega)I + \omega U], \tag{1.4}$$

and

$$L = D^{-1}A_L, \qquad U = D^{-1}A_U. \tag{1.5}$$

Now (1.3) converges if and only if the spectral radius of $\mathscr{L}_\omega$ is less than unity, and the asymptotic rate of convergence is given by

$$R_\infty(\mathscr{L}_\omega) = -\log(\rho(\mathscr{L}_\omega)). \tag{1.6}$$

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The SOR method has been extensively studied for a symmetric positive definite matrix A (see, e.g., Varga [5] and Young [7]). For a positive definite and consistently ordered matrix A, from [5] and [7] we have:

S1. $\rho(\mathscr{L}_\omega) < 1 \Leftrightarrow \bar{\mu} < 1$ and $0 < \omega < 2$.

S2.

$$\rho(\mathscr{L}_\omega) = \begin{cases} \{\omega\bar{\mu} + [\omega^2\bar{\mu}^2 - 4(\omega-1)^{1/2}]^{1/2}\}^2/4 & \text{if } 0 < \omega < \omega_b', \\ \omega - 1 & \text{if } \omega_b' \le \omega < 2. \end{cases}$$

S3. $\rho(\mathscr{L}_{\omega_b'}) < \rho(\mathscr{L}_\omega)$ if $\omega \ne \omega_b'$.

Here,

$$\bar{\mu} = \rho(B) = \rho(L+U), \qquad \omega_b' = 2/[1 + (1-\bar{\mu}^2)^{1/2}].$$

Young [6] has shown that if A is consistently ordered and the eigenvalues of B are real and less than unity in modulus, then the Jordan canonical form of $\mathscr{L}_{\omega_b'}$ is not diagonal. Therefore, in this case, the SOR method converges slower than expected based on the spectral radius $\rho(\mathscr{L}_{\omega_b'})$. When A is symmetric and positive definite, and when A has the form

$$A = \begin{pmatrix} D_1 & -H \\ -K & D_2 \end{pmatrix}, \tag{1.7}$$

Young [7, Chapter 7] determined the $D^{1/2}$-norm and $A^{1/2}$-norm of $\mathscr{L}_{\omega_b'}$ (the spectral norms of $D^{1/2}\mathscr{L}_{\omega_b'}D^{-1/2}$ and $A^{1/2}\mathscr{L}_{\omega_b'}A^{-1/2}$, respectively) and pointed out that the $D^{1/2}$-norm of $\mathscr{L}_{\omega_b'}$ is greater than unity in general. Moreover, $\|\mathscr{L}_{\omega_b'}^m\|$ (the spectral norm of $\mathscr{L}_{\omega_b'}^m$) behaves much like $\|\mathscr{L}_{\omega_b'}^m\|_{D^{1/2}}$. However, for large m, $\|\mathscr{L}_{\omega_b'}^m\|_{D^{1/2}} < 1$, and eventually $\|\mathscr{L}_{\omega_b'}^m\|_{D^{1/2}}$ tends to zero, though considerably more slowly than $\rho(\mathscr{L}_{\omega_b'}^m)$.

For the matrix A in (1.1) the unsymmetric case is by far not as common as the symmetric one, but nevertheless, unsymmetric matrices appear, e.g., in the numerical solution of the biharmonic equation [1] and the computation of cubic splines, [3] and [4, Chapter 3]. If the matrix A (1.2) is consistently ordered and B, given by

$$B = L + U, \tag{1.8}$$

is similar to a skew-symmetric matrix and has either zero eigenvalues or purely imaginary eigenvalues, then from [1], [3], and [4], or the theory of Young [7], we have:

US1. $\rho(\mathscr{L}_\omega) < 1 \Leftrightarrow 0 < \omega < 2/(1+\bar{\mu})$.

US2.

$$\rho(\mathscr{L}_\omega) = \begin{cases} 1 - \omega & \text{if } 0 < \omega \le \omega_b, \\ \left[\dfrac{\bar{\mu}\omega + \sqrt{\omega^2\bar{\mu}^2 + 4(\omega-1)}}{2}\right]^2 & \text{if } \omega_b < \omega < \dfrac{2}{1+\bar{\mu}}. \end{cases}$$

US3. $\rho(\mathscr{L}_{\omega_b}) < \rho(\mathscr{L}_\omega)$ if $\omega \ne \omega_b$.

Here,

$$\bar{\mu} = \rho(B), \qquad \omega_b = 2/(1 + \sqrt{1+\bar{\mu}^2}). \tag{1.9}$$

Notice that in this case we can always choose the relaxation factor ω such that $\rho(\mathscr{L}_\omega) < 1$, no matter how large $\bar{\mu}$ is. This is very different from the symmetric case. Another difference between the two cases is that the optimum factor ω_b for the unsymmetric case is less than unity and the optimum factor ω_b' for the symmetric case is greater than unity. It is also important to note that overestimating ω_b' is better than an underestimation, but for ω_b an underestimate is better than overestimating.

However, to our knowledge, the Jordan canonical form and $D^{1/2}$-norm of $\mathscr{L}_\omega$ for the unsymmetric case are not discussed in the literature.

In this paper we will investigate these problems under the assumption that in (1.1) the matrix A has the special form (1.7) with D_1 and D_2 symmetric and positive definite and $K^T = -H$. We will obtain some results similar to those for the symmetric case.

In the next section we review some properties for skew-symmetric matrices required for their application in the later sections. In Section 3 we construct the basis of eigenvectors of the associated Jacobi matrix B which is similar to a skew-symmetric matrix. In Section 4 we will show that the Jordan canonical form of $\mathscr{L}_{\omega_b}$ is not a diagonal matrix, but has only q principal vectors of grade 2 associated with $\omega_b - 1$, the eigenvalues of $\mathscr{L}_{\omega_b}$. Here, q is the multiplicity of the eigenvalue $i\bar{\mu}$ $(= i\rho(B))$ of B. Hence, $\|\mathscr{L}_{\omega_b}^m\|$, the spectral norm, behaves like $m \cdot \rho(\mathscr{L}_{\omega_b})^{m-1}$ rather than $\rho(\mathscr{L}_{\omega_b})^m$.

In Section 5 we will determine the $D^{1/2}$-norm of $\mathscr{L}_\omega$ and point out that if $\bar{\mu} = \rho(B) \geq 1$, then $\|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}} \geq 1$. However, in Section 6, we will show that for any $\bar{\mu} > 0$, for m large enough, $\|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}} < 1$. Eventually, $\|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}}$ converges to zero, though considerably more slowly than $\rho(\mathscr{L}_{\omega_b}^m)$.

In this paper, almost all the notations used are the same as those adopted by Young [7], and all our work is based on the theory of Young [7].

2. Some Properties of Skew-Symmetric Matrices. Let $A \in R^{n\times n}$ and

$$A^T = -A. \tag{2.1}$$

It is well known that A has the following properties:

(a) All diagonal elements of A are zero.

(b) A has either zero eigenvalues or purely imaginary eigenvalues, that is, any eigenvalue μ of A has the form

$$\mu = i\xi. \tag{2.2}$$

Here ξ is real. Also, $-\mu = -i\xi$ is an eigenvalue of A.

(c) A is a normal matrix, that is,

$$A^T A = AA^T. \tag{2.3}$$

(d) A is unitarily similar to a diagonal matrix.

All the above properties are easy to prove and can be found in any textbook of linear algebra, e.g., see [2].

3. The Eigenvectors of the Associated Jacobi Iteration Matrix B. Consider A in (1.1) to have the special form

$$A = \begin{pmatrix} D_1 & -H \\ -K & D_2 \end{pmatrix}, \tag{3.1}$$

where D_1 ($\in R^{r\times r}$) and D_2 ($\in R^{s\times s}$) are symmetric positive definite and

$$H^T = -K; \tag{3.2}$$

the associated Jacobi iterative matrix B has the form

$$B = \begin{pmatrix} 0 & F \\ G & 0 \end{pmatrix}. \tag{3.3}$$

Here,

$$G = D_2^{-1}K, \qquad F = D_1^{-1}H. \tag{3.4}$$

Because D_1 and D_2 are positive definite, we can choose symmetric and positive definite matrices $D_1^{1/2}$ and $D_2^{1/2}$ such that

$$D_1^{1/2}D_1^{1/2} = D_1, \qquad D_2^{1/2}D_2^{1/2} = D_2. \tag{3.5}$$

If we write

$$D = \begin{pmatrix} D_1^{1/2} & 0 \\ 0 & D_2^{1/2} \end{pmatrix}, \tag{3.6}$$

then we have

$$D^{1/2}BD^{-1/2} = \begin{pmatrix} 0 & D_1^{-1/2}HD_2^{-1/2} \\ D_2^{-1/2}KD_1^{-1/2} & 0 \end{pmatrix}. \tag{3.7}$$

Hence B is similar to a skew-symmetric matrix, and thus unitarily similar to a diagonal matrix.

In this section we will construct a basis of eigenvectors for B. From (3.3) we have

$$B^2 = \begin{pmatrix} FG & 0 \\ 0 & GF \end{pmatrix}. \tag{3.8}$$

Evidently, B^2 is also similar to a diagonal matrix and, in fact, the $(r\times r)$ matrix FG and the $(s\times s)$ matrix GF are also similar to diagonal matrices, where $r+s=n$, the order of the matrix A. Also note that FG and GF have nonpositive eigenvalues. Let the p eigenvectors of FG associated with the nonzero eigenvalues $\nu_1, \nu_2, \dots, \nu_p$ be $\xi_1, \xi_2, \dots, \xi_p$, i.e.,

$$FG\xi_j = \nu_j\xi_j, \qquad j = 1, 2, \dots, p. \tag{3.9}$$

If we let

$$\eta_j = G\xi_j, \qquad j = 1, 2, \dots, p, \tag{3.10}$$

then $\eta_j \neq 0$, and η_j is an eigenvector of GF associated with ν_j, i.e.,

$$GF\eta_j = \nu_j\eta_j, \qquad j = 1, 2, \dots, p. \tag{3.11}$$

Moreover, since the ξ_j, $j = 1, 2, \dots, p$, are linearly independent, then so are the η_j, $j = 1, 2, \dots, p$, since

$$\sum_{j=1}^{p} c_j\eta_j = 0$$

implies that

$$0 = F\left(\sum_{j=1}^{p} c_j\eta_j\right) = \sum_{j=1}^{p} \nu_j c_j \xi_j = 0,$$

and hence the c_j, $j = 1, 2, \ldots, p$, vanish because of the linear independence of the ξ_j, $j = 1, 2, \ldots, p$. Evidently, there can be no more than p eigenvectors of GF associated with nonzero eigenvalues; otherwise, there would be more than p linearly independent eigenvectors of FG associated with the nonzero eigenvalues. Thus, we have

$$p \le \min\{r, s\}. \tag{3.12}$$

Since $\nu_j < 0$, $j = 1, 2, \ldots, p$, if we let

$$\mu_j = i|\nu_j|^{1/2}, \quad x_j = \mu_j \xi_j, \quad y_j = \eta_j, \quad v_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix}, \qquad j = 1, 2, \ldots, p, \tag{3.13}$$

where $i^2 = -1$, then using (3.9), (3.11) and (3.13), we have

$$Bv_j = \begin{pmatrix} Fy_j \\ Gx_j \end{pmatrix} = \begin{pmatrix} \nu_j \xi_j \\ \mu_j \eta_j \end{pmatrix} = \begin{pmatrix} \mu_j \mu_j \xi_j \\ \mu_j \eta_j \end{pmatrix} = \mu_j v_j, \qquad j = 1, 2, \ldots, p. \tag{3.14}$$

Notice that μ_j, $j = 1, 2, \ldots, p$, have positive imaginary parts.

Let us now define for $j = p+1,\ p+2, \ldots, 2p$

$$x_j = x_{j-p}, \quad y_j = -y_{j-p}, \quad v_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix}, \quad \mu_j = -\mu_{j-p}. \tag{3.15}$$

Evidently, we have

$$Bv_j = \mu_j v_j, \qquad j = p+1, p+2, \ldots, 2p. \tag{3.16}$$

If we let $FGx = 0$, where $x \ne 0$, then by (3.4) we have

$$D_1^{-1} H D_2^{-1} K x = 0, \quad \text{or} \quad H D_2^{-1} K x = 0.$$

Thus, we have

$$-HD_2^{-1/2} D_2^{-1/2} H^T x = 0 \quad \text{or} \quad (D_2^{-1/2} H^T x)^* (D_2^{-1/2} H^T x) = 0. \tag{3.17}$$

Here * stands for the conjugate transpose of a matrix. Hence from (3.17) we have $D_2^{-1/2} H^T x = 0$, or $D_2^{-1} H^T x = Gx = 0$. Therefore, we have that if $FGx = 0$, where $x \ne 0$, then

$$B\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Gx \end{pmatrix} = 0. \tag{3.18}$$

Thus, if the eigenvectors of FG associated with the eigenvalue zero are x_{2p+1}, $x_{2p+2}, \ldots, x_{p+r}$, then the vectors

$$v_j = \begin{pmatrix} x_j \\ 0 \end{pmatrix}, \qquad j = 2p+1, 2p+2, \ldots, p+r, \tag{3.19}$$

are eigenvectors of B associated with the eigenvalue zero. Similarly, if the eigenvectors of GF associated with the eigenvalue zero are $y_{p+r+1}, y_{p+r+2}, \ldots, y_{s+r}$, then the vectors

$$v_j = \begin{pmatrix} 0 \\ y_j \end{pmatrix}, \qquad j = p+r+1,\ p+r+2, \ldots, n = r+s, \tag{3.20}$$

are eigenvectors of B associated with the eigenvalue zero.

We have thus constructed a basis of eigenvectors for B,

$$v_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix}, \qquad j = 1, 2, \ldots, n, \tag{3.21}$$

and moreover, we have by (3.14) and (3.16)

$$Gx_j = \mu_j y_j, \quad Fy_j = \mu_j x_j, \qquad j = 1, 2, \ldots, n. \tag{3.22}$$

We also have

$$\text{for } \mu_j = i|\nu_j|^{1/2}, \quad v_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix}, \qquad j = 1, 2, \ldots, p;$$
$$\text{for } \mu_j = -i|\nu_j|^{1/2}, \quad v_j = \begin{pmatrix} x_{j-p} \\ -y_{j-p} \end{pmatrix}, \qquad j = p+1, \ldots, 2p;$$
$$\text{for } \mu_j = 0, \quad v_j = \begin{pmatrix} x_j \\ 0 \end{pmatrix}, \qquad j = 2p+1, \ldots, r+p;$$
$$\text{for } \mu_j = 0, \quad v_j = \begin{pmatrix} 0 \\ y_j \end{pmatrix}, \qquad j = p+r+1, \ldots, n.$$

4. The Principal Vectors of $\mathscr{L}_\omega$. We now seek the eigenvectors and principal vectors of $\mathscr{L}_\omega$ for $\omega \neq 0$. Because A has the form of (3.1), from (3.3) we have

$$L = \begin{bmatrix} 0 & 0 \\ G & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}. \tag{4.1}$$

Thus we have

$$\begin{aligned} \mathscr{L}_\omega &= (I - \omega L)^{-1}((1-\omega)I + \omega U) \\ &= \begin{bmatrix} I_1 & 0 \\ -\omega G & I_2 \end{bmatrix}^{-1} \begin{bmatrix} (1-\omega)I_1 & \omega F \\ 0 & (1-\omega)I_2 \end{bmatrix} \\ &= \begin{bmatrix} I_1 & 0 \\ \omega G & I_2 \end{bmatrix} \begin{bmatrix} (1-\omega)I_1 & \omega F \\ 0 & (1-\omega)I_2 \end{bmatrix} \\ &= \begin{bmatrix} (1-\omega)I_1 & \omega F \\ \omega(1-\omega)G & \omega^2 GF + (1-\omega)I_2 \end{bmatrix}, \end{aligned} \tag{4.2}$$

where I_1 and I_2 are identity matrices of the same sizes as D_1 and D_2, respectively. For each nonzero eigenvalue μ of B, let $\lambda_+^{1/2}$ and $\lambda_-^{1/2}$ be the roots of

$$\lambda + \omega - 1 = \omega\mu\lambda^{1/2}. \tag{4.3}$$

Since

$$B\begin{pmatrix} x \\ y \end{pmatrix} = \mu \begin{pmatrix} x \\ y \end{pmatrix},$$

the vectors

$$w = \begin{pmatrix} x \\ \lambda_+^{1/2} y \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ \lambda_-^{1/2} y \end{pmatrix} \tag{4.4}$$

are the eigenvectors of $\mathscr{L}_\omega$, since by (4.2), (3.22) and (4.3) we have

$$\begin{aligned} \mathscr{L}_\omega \begin{pmatrix} x \\ \lambda_+^{1/2} y \end{pmatrix} &= \begin{pmatrix} (1 - \omega + \omega\mu\lambda_+^{1/2})x \\ [\omega\mu(1 - \omega + \omega\mu\lambda_+^{1/2}) + (1-\omega)\lambda_+^{1/2}]y \end{pmatrix} \\ &= \lambda_+ w \end{aligned} \tag{4.5}$$

and

$$\mathscr{L}_\omega z = \lambda_- z. \tag{4.6}$$

If we let

$$v = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \hat{v} = \begin{pmatrix} x \\ -y \end{pmatrix}, \tag{4.7}$$

then we have

$$\begin{aligned} w &= \tfrac{1}{2}(v+\hat{v}) + \tfrac{1}{2}\lambda_+^{1/2}(v-\hat{v}) = \tfrac{1}{2}(1+\lambda_+^{1/2})v + \tfrac{1}{2}(1-\lambda_+^{1/2})\hat{v}, \\ z &= \tfrac{1}{2}(v+\hat{v}) + \tfrac{1}{2}\lambda_-^{1/2}(v-\hat{v}) = \tfrac{1}{2}(1+\lambda_-^{1/2})v + \tfrac{1}{2}(1-\lambda_-^{1/2})\hat{v}. \end{aligned} \tag{4.8}$$

If $\lambda_-^{1/2} \neq \lambda_+^{1/2}$, then w and z are linearly independent. But for $\omega\mu \neq 0$, the discriminant $\omega^2\mu^2 - 4(\omega - 1)$ of (4.3) does not vanish unless

$$\omega^2|\mu|^2 + 4(\omega - 1) = 0. \tag{4.9}$$

On the other hand, if (4.9) holds and if $\omega\mu \neq 0$, then $\lambda_+^{1/2} = \lambda_-^{1/2} = \lambda^{1/2} = \omega\mu/2 \neq 0$, and w and z are not linearly independent. Notice that in this case,

$$\lambda_+ = \lambda_- = \lambda = \omega^2\mu^2/4 = \omega - 1. \tag{4.10}$$

If we let

$$\hat{z} = \frac{1}{2}\cdot\frac{1}{\lambda^{1/2}}\begin{pmatrix} 0 \\ y \end{pmatrix}, \tag{4.11}$$

then we have

$$\begin{aligned} \mathscr{L}_\omega \hat{z} &= \begin{pmatrix} (1-\omega)I_1 & \omega F \\ \omega(1-\omega)G & \omega^2 GF + (1-\omega)I_2 \end{pmatrix} \cdot \frac{1}{2\lambda^{1/2}}\begin{pmatrix} 0 \\ y \end{pmatrix} \\ &= \frac{1}{2\lambda^{1/2}}\begin{pmatrix} \omega F y \\ \omega^2 GFy + (1-\omega)y \end{pmatrix} = \frac{1}{2\lambda^{1/2}}\begin{pmatrix} \omega\mu x \\ \omega\mu^2 y + (1-\omega)y \end{pmatrix} \\ &= \frac{1}{2\lambda^{1/2}}\begin{pmatrix} \omega\mu x \\ [\omega^2\mu^2 + 1 - \omega - \lambda + \lambda]y \end{pmatrix} \\ &= \frac{1}{2\lambda^{1/2}}\begin{pmatrix} 2\cdot\dfrac{\omega\mu}{2}x \\ [\omega^2\mu^2 + 1 - \omega - (\omega-1)]y \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ \dfrac{1}{2\lambda^{1/2}}y \end{pmatrix} \\ &= \frac{1}{2\lambda^{1/2}}\begin{pmatrix} 2\cdot\lambda^{1/2}x \\ [4(\omega-1) + 2(1-\omega)]y \end{pmatrix} + \lambda\hat{z} = \frac{1}{2\lambda^{1/2}}\begin{pmatrix} 2\lambda^{1/2}x \\ 2(\omega-1)y \end{pmatrix} + \lambda\hat{z} \\ &= \begin{pmatrix} x \\ \lambda^{1/2}y \end{pmatrix} + \lambda\hat{z} = w + \lambda\hat{z}. \end{aligned} \tag{4.12}$$

Hence, $\hat{z}$ is a principal vector of grade 2. Moreover, we have

$$w = \frac{1}{2}(1+\lambda^{1/2})v + \frac{1}{2}(1-\lambda^{1/2})\hat{v}, \qquad \hat{z} = \frac{1}{4}\cdot\frac{1}{\lambda^{1/2}}v - \frac{1}{4}\cdot\frac{1}{\lambda^{1/2}}\hat{v}. \tag{4.13}$$

Thus w and $\hat{z}$ are linearly independent.

If we let, for $\omega\mu \neq 0$,

$$\begin{aligned} w_j &= \begin{pmatrix} x_j \\ (\lambda_j)_+^{1/2} y_j \end{pmatrix}, \qquad j = 1, 2, \ldots, p; \\ w_{j+p} &= \begin{cases} \begin{pmatrix} x_j \\ (\lambda_j)_-^{1/2} y_j \end{pmatrix}, & j = 1, 2, \ldots, p, \ \text{if } \omega^2|\mu_j|^2 + 4(\omega-1) \neq 0, \\ \dfrac{1}{2}\cdot\dfrac{1}{(\lambda_j)_+^{1/2}}\begin{pmatrix} 0 \\ y_j \end{pmatrix}, & j = 1, 2, \ldots, p, \ \text{if } \omega^2|\mu_j|^2 + 4(\omega-1) = 0; \end{cases} \\ w_j &= v_j, \qquad j = 2p+1, 2p+2, \ldots, n, \end{aligned} \tag{4.14}$$

then we can easily prove that w_j, $j = 1, 2, \ldots, n$, are linearly independent and hence form a basis of the n-dimensional complex vector space $\mathbf{C}^n$. Therefore, the matrix whose columns are the w_j reduces $\mathcal{L}_\omega$ to Jordan canonical form. Thus we have proved:

THEOREM 1. *If the matrix A has the form of* (3.1) *and $i\bar{\mu} = i\rho(B)$ is an eigenvalue of multiplicity q of B of* (3.3), *then the Jordan canonical form of $\mathcal{L}_{\omega_b}$ has $n-2q$ (1×1) sub-Jordan blocks and q (2×2) sub-Jordan blocks which correspond to the eigenvalue $\omega_b - 1$.*

Notice that if $q = 1$, then the Jordan canonical form of $\mathcal{L}_{\omega_b}$ has one nondiagonal element.

From Theorem 3.1 of [5, p. 65] and Theorem 3-7.1 of [7, p. 85] we have

$$\|\mathcal{L}_{\omega_b}^m\| \sim J(\mathcal{L}_{\omega_b}) \cdot m \cdot \rho(\mathcal{L}_{\omega_b})^{m-1}. \tag{4.15}$$

Here, $\|A\|$ is the spectral norm of the matrix A and $J(\mathcal{L}_{\omega_b})$ is the Jordan condition number of the matrix $\mathcal{L}_{\omega_b}$, defined by Young [7, p. 85] and given by

$$J(\mathcal{L}_{\omega_b}) = \inf_{V \in S_1} \kappa(V), \tag{4.16}$$

where $\kappa(V)$ is the spectral condition number of the matrix V and S_1 the set of all matrices such that

$$V^{-1}\mathcal{L}_{\omega_b}V = J. \tag{4.17}$$

Here, J is the Jordan canonical form of the matrix $\mathcal{L}_{\omega_b}$.

5. Determination of $\|\mathcal{L}_\omega\|_{D^{1/2}}$. Let

$$\hat{\mathcal{L}}_\omega = D^{1/2}\mathcal{L}_\omega D^{-1/2}; \tag{5.1}$$

then from (4.2) and (3.6) we have

$$\hat{\mathcal{L}}_\omega = \begin{pmatrix} (1-\omega)I_1 & \omega D_1^{1/2} F D_2^{-1/2} \\ \omega(1-\omega)D_2^{1/2} G D_1^{-1/2} & \omega^2 D_2^{1/2} G F D_2^{-1/2} + (1-\omega)I_2 \end{pmatrix}. \tag{5.2}$$

If we let

$$\begin{aligned} \hat{F} &= D_1^{1/2} F D_2^{-1/2} = D_1^{-1/2} H D_2^{-1/2}, \\ \hat{G} &= D_2^{1/2} G D_1^{-1/2} = D_2^{-1/2} K D_1^{-1/2}, \end{aligned} \tag{5.3}$$

then

$$\hat{G}^T = -\hat{F} \tag{5.4}$$

and

$$\hat{\mathcal{L}}_\omega = \begin{pmatrix} (1-\omega)I_1 & \omega\hat{F} \\ \omega(1-\omega)\hat{G} & \omega^2\hat{G}\hat{F} + (1-\omega)I_2 \end{pmatrix}. \tag{5.5}$$

Hence, $\hat{\mathcal{L}}_\omega$ is the SOR iterative matrix corresponding to the matrix

$$\hat{A} = D^{1/2} A D^{-1/2} = \begin{pmatrix} I_1 & -\hat{F} \\ -\hat{G} & I_2 \end{pmatrix} \tag{5.6}$$

with the associated Jacobi iterative matrix

$$\hat{B} = \begin{pmatrix} 0 & \hat{F} \\ \hat{G} & 0 \end{pmatrix} = D^{1/2} B D^{-1/2}. \tag{5.7}$$

Therefore, $\rho(\hat{B}) = \rho(B)$, and ω_b is the same for $\hat{A}$ as for A. Moreover, if we let $\mathscr{L}_\omega[A]$ stand for the SOR iterative matrix associated with the matrix A and $D[A]$ for the diagonal block of the matrix A, then we have

$$\|\mathscr{L}_\omega^m[A]\|_{D[A]^{1/2}} = \|\hat{\mathscr{L}}_\omega^m[A]\| = \|\mathscr{L}_\omega^m[\hat{A}]\| = \|\mathscr{L}_\omega^m[\hat{A}]\|_{D[\hat{A}]^{1/2}}. \tag{5.8}$$

Thus, it is sufficient to assume A of (3.1) with $D_1 = I_1$ and $D_2 = I_2$. Otherwise, we consider $\hat{A}$ (5.6). Notice that when $D_1 = I_1$ and $D_2 = I_2$ then $F = H$, $G = K$, and $F^T = -G$.

Since

$$\|\mathscr{L}_\omega^T\|_{D^{1/2}} = \|\mathscr{L}_\omega\|_{D^{1/2}} = [\rho(\mathscr{L}_\omega\mathscr{L}_\omega^*)]^{1/2}, \tag{5.9}$$

according to the expression (4.2) for $\mathscr{L}_\omega$ we first study the eigenvalues of products of matrices of the form

$$\begin{pmatrix} a_{11}(FG) & a_{12}(FG)F \\ a_{21}(GF)G & a_{22}(GF) \end{pmatrix}, \tag{5.10}$$

where a_{11} and a_{12} are polynomials in FG and a_{21} and a_{22} polynomials in GF. By an analogy with Theorem 7-2.1 of Young [7, p. 239] we have:

THEOREM 2. *If B is a matrix of the form* (3.3), *then*

(a) *The matrix*

$$Q = \begin{pmatrix} a_{11}(FG) & a_{12}(FG)F \\ a_{21}(GF)G & a_{22}(GF) \end{pmatrix} \tag{5.11}$$

is nonsingular if

$$\tau(B^2) = a_{11}(B^2)a_{22}(B^2) - a_{21}(B^2)a_{12}(B^2)B^2 \tag{5.12}$$

in nonsingular. Moreover, $\tau(B^2)$ is nonsingular if and only if for each eigenvalue μ of B the matrix

$$R(\mu) = \begin{pmatrix} a_{11}(\mu^2) & a_{12}(\mu^2)\mu \\ a_{21}(\mu^2)\mu & a_{22}(\mu^2) \end{pmatrix} \tag{5.13}$$

is nonsingular.

(b) *Let*

$$G_m = \prod_{k=m}^{1} \begin{pmatrix} a_{11}^{(k)}(FG) & a_{12}^{(k)}(FG)F \\ a_{21}^{(k)}(GF)G & a_{22}^{k}(GF) \end{pmatrix}^{\nu_k}, \tag{5.14}$$

where for each k, $\nu_k = \pm 1$. It is assumed that for any k the matrix

$$\tau^{(k)}(B^2) = a_{11}^{(k)}(B^2)a_{22}^{(k)}(B^2) - a_{21}^{(k)}(B^2)a_{12}^{(k)}(B^2)B^2 \tag{5.15}$$

is nonsingular for $\nu_k = -1$. For each eigenvalue μ of B, let

$$M_m(\mu) = \prod_{k=m}^{1} \begin{pmatrix} a_{11}^{(k)}(\mu^2) & a_{12}^{(k)}(\mu^2)\mu \\ a_{21}^{(k)}(\mu^2)\mu & a_{22}^{(k)}(\mu^2) \end{pmatrix}^{\nu_k}. \tag{5.16}$$

If μ is a nonzero eigenvalue of B and if λ is an eigenvalue of $M_m(\mu)$, then λ is an eigenvalue of G_m. If $\mu = 0$ is an eigenvalue of B, then at least one of the eigenvalues of $M_m(0)$ is an eigenvalue of G_m.

(c) *If λ is an eigenvalue of G_m, then there exists an eigenvalue μ of B such that λ is an eigenvalue of $M_m(\mu)$.*

Notice that although the matrix B considered here and the matrix B considered in Theorem 7-2.1 of Young [7, p. 239] are not the same type of matrices—the former is similar to a skew-symmetric matrix and the latter a symmetric matrix—the statement of these two theorems are the same and the proofs are also the same. Hence the proof of Theorem 2 is omitted.

From (4.2) we have

$$\tag{5.17} \begin{aligned} \mathscr{L}_\omega^* = \mathscr{L}_\omega^T &= \begin{pmatrix} (1-\omega)I_1 & 0 \\ -\omega G & (1-\omega)I_2 \end{pmatrix} \begin{pmatrix} I_1 & -\omega F \\ 0 & I_2 \end{pmatrix} \\ &= \begin{pmatrix} (1-\omega)I_1 & -\omega(1-\omega)F \\ -\omega G & \omega^2 GF + (1-\omega)I_2 \end{pmatrix}. \end{aligned}$$

Thus, from (4.2) and (5.7), $\mathscr{L}_\omega$ and $\mathscr{L}_\omega^*$ have the required form for the applicability of Theorem 2. From Theorem 2 we know that the eigenvalues of $\mathscr{L}_\omega \mathscr{L}_\omega^*$ are the same as the eigenvalues of $M(\omega,\mu)M^*(\omega,\mu)$, where

$$\tag{5.18} \begin{aligned} M(\omega,\mu) &= \begin{pmatrix} 1 & 0 \\ \omega\mu & 1 \end{pmatrix} \begin{pmatrix} 1-\omega & \omega\mu \\ 0 & 1-\omega \end{pmatrix} \\ &= \begin{pmatrix} 1-\omega & \omega\mu \\ (1-\omega)\omega\mu & \omega^2\mu^2 + 1 - \omega \end{pmatrix}. \end{aligned}$$

If we notice $\bar{\mu} = -\mu$ (here μ is purely imaginary), and if we let

$$\tag{5.19} M(\omega,\mu)M^*(\omega,\mu) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

then

$$\begin{aligned} m_{11} &= (1-\omega)^2 - \omega^2\mu^2, \\ m_{12} &= \omega^3\mu^3 + \omega\mu(1-\omega) - (1-\omega)^2 \cdot \omega\mu = \mu\omega^2[1-\omega+\omega\mu^2], \\ m_{21} &= -m_{12} = -\mu\omega^2[1-\omega+\omega\mu^2], \\ m_{22} &= \omega^4\mu^4 + (1-\omega)^2 + 2\omega^2\mu^2(1-\omega) - \omega^2\mu^2(1-\omega)^2 \\ &= \omega^4\mu^4 + (1-\omega)^2 + \omega^2\mu^2(1-\omega)(1+\omega). \end{aligned}$$

Since

$$\tag{5.20} \begin{aligned} m_{11} + m_{22} &= 2(1-\omega)^2 + \omega^4\mu^4 - \omega^4\mu^2, \\ m_{12}m_{21} &= -\mu^2\omega^4[(1-\omega)^2 + 2\omega\mu^2(1-\omega) + \omega^2\mu^4], \\ m_{22}m_{11} &= -\omega^6\mu^6 - 2\omega^4\mu^4 \cdot \omega \cdot (1-\omega) - \omega^2\mu^2(1-\omega)^2 \cdot \omega^2 + (1-\omega)^4 \\ &= -\omega^4\mu^2[\omega^2\mu^4 + 2\omega\mu^2(1-\omega) + (1-\omega)^2] + (1-\omega)^4 \\ &= m_{21}m_{12} + (1-\omega)^4, \end{aligned}$$

we have

$$\tag{5.21} m_{22}m_{11} - m_{21}m_{12} = (1-\omega)^4.$$

Thus, if we let

$$\tag{5.22} \det(\lambda I - M(\omega,\mu)M^*(\omega,\mu)) = \lambda^2 - T(\mu^2)\lambda + c = 0,$$

then

$$T(\mu^2) = 2(1-\omega)^2 + \omega^4\mu^4 - \omega^4\mu^2, \qquad c = (1-\omega)^4. \tag{5.23}$$

Notice that $\mu^2 \le 0$, so that $T(\mu^2)$ is an increasing function of $|\mu|$. Therefore, by Lemma 6-2.9 of Young [7, p. 186], it follows that for a given ω, the largest value of the root radius of (5.22) is assumed for $\mu = i\rho(B)$ (or $\mu = -i\rho(B)$). From (5.22) and (5.23) we know that if λ satisfies (5.22), then $t = \lambda^{1/2}$ satisfies

$$t^2 - (\omega-1)^2 = \omega^2|\mu|(1+|\mu|^2)^{1/2}t. \tag{5.24}$$

Note $|\mu| = \rho(B)$, and if we let $\bar{\mu} = \rho(B)$ and

$$d = \bar{\mu}(1+\bar{\mu}^2)^{1/2}, \tag{5.25}$$

then by Lemma 6-2.1 of Young [7, p. 171] the root radius of (5.24) is less than unity if and only if we have

$$|\omega - 1| < 1 \quad \text{and} \quad \omega^2 d < 1 - (\omega-1)^2 = \omega(2-\omega),$$

or, equivalently,

$$0 < \omega < \min\left\{2, \frac{2}{1+d}\right\} = \frac{2}{1+d} \quad (\bar{\mu} > 0). \tag{5.26}$$

Thus we have proved

THEOREM 3. *If A has the form* (3.1) *with D_1 and D_2 symmetric and positive definite and H and K satisfying* (3.2), *then $\|\mathscr{L}_\omega\|_{D^{1/2}} < 1$ if and only if ω satisfies* (5.26). *Moreover, we have*

$$\|\mathscr{L}_\omega\|_{D^{1/2}} = \frac{\omega^2 d + \sqrt{\omega^4 d^2 + 4(1-\omega)^2}}{2}. \tag{5.27}$$

We now determine the minimum value of $\|\mathscr{L}_\omega\|_{D^{1/2}}$. If we let

$$f(\omega) = \omega^2 d + \sqrt{\omega^4 d^2 + 4(1-\omega)^2},$$

then the derivative of $f(\omega)$ is given by

$$f'(\omega) = 2\omega d + [4\omega^3 d^2 + 8(\omega-1)]/2\sqrt{\omega^4 d^2 + 4(1-\omega)^2}.$$

Assume $f'(\omega) = 0$; then

$$-\omega d\sqrt{\omega^4 d^2 + 4(1-\omega)^2} = \omega^3 d^2 + 2(\omega - 1). \tag{5.28}$$

Notice that (5.28) means

$$g(\omega) = \omega^3 d^2 + 2(\omega-1) < 0. \tag{5.29}$$

By Descartes' rule we know that $g(\omega)$ has only one positive root ω_u. Thus, if $\omega \in (0, \omega_u)$, then (5.29) holds. Moreover, if $\omega \ge \omega_u$, we have

$$f'(\omega) > 0. \tag{5.30}$$

If $0 < \omega < \omega_u$, and from (5.28), we have

$$\omega^2 d^2 + \omega - 1 = 0. \tag{5.31}$$

Evidently, the positive root ω_+ of (5.31) is given by

$$\omega_+ = [-1 + \sqrt{1+4d^2}]/2d^2 = \frac{2}{1+\sqrt{1+4d^2}}. \tag{5.32}$$

One can examine

$$\omega_+ < \min\left\{2, \frac{2}{1+d}\right\} = \frac{2}{1+d}. \tag{5.33}$$

Thus, we obtain

$$f'(\omega)\begin{cases} < 0 & \text{if } 0 < \omega < \omega_+, \\ = 0 & \text{if } \omega = \omega_+, \\ > 0 & \text{if } \omega > \omega_+, \end{cases}$$

because we have

$$\omega_+ < \omega_u. \tag{5.34}$$

In fact, if $\omega_+ \geq \omega_u$, then from (5.31) and (5.32) we have $\omega^2 d^2 + \omega - 1 < 0$ for $0 < \omega < \omega_u$. Thus we can prove $f'(\omega) < 0$ for $0 < \omega < \omega_u$. Owing to the continuity property of $f'(\omega)$, we have $f'(\omega_u) \leq 0$, which contradicts (5.30). Hence (5.34) holds. We have now proved the following theorem.

THEOREM 4. *Under the assumptions of Theorem* 3 *we have*

$$\|\mathscr{L}_{\omega_+}\|_{D^{1/2}} < \|\mathscr{L}_\omega\|_{D^{1/2}} \quad \textit{for } \omega \neq \omega_+.$$

Here, ω_+ *is given by* (5.32).

It is important to note that from (1.9), (5.25), and (5.26) we have

$$\omega_b < \frac{2}{1+d} \quad \text{if } \bar{\mu} = \rho(B) < 1,$$
$$\omega_b \geq \frac{2}{1+d} \quad \text{if } \bar{\mu} \geq 1.$$

Thus, when $\bar{\mu} < 1$, we also have $\|\mathscr{L}_{\omega_b}\|_{D^{1/2}} < 1$.

In fact we have proved

THEOREM 5. *Under the assumptions of Theorem* 3 *we have*

$$\|\mathscr{L}_{\omega_b}\|_{D^{1/2}} < 1 \quad \textit{if and only if } \bar{\mu} = \rho(B) < 1.$$

But when $\bar{\mu} \geq 1$, we have $\|\mathscr{L}_{\omega_b}\|_{D^{1/2}} \geq 1$. However, in the next section, we will prove that for any $\bar{\mu} > 0$, $\|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}} < 1$ if m is large enough, and that $\text{limit}_{m\to\infty}\|\mathscr{L}_{\omega_b}^m\| = 0$.

6. Determination of $\|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}}$. In this section we continue with the theory of Young [7, Chapter 7] to investigate $\|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}}$. From the discussion of the last section it is sufficient to consider A (3.1) with $D_1 = I_1$ and $D_2 = I_2$. Since the eigenvalues of $\mathscr{L}_{\omega_b}^m(\mathscr{L}_{\omega_b}^m)^*$ are the same as those of $M^m(\omega_b,\mu)[M^m(\omega_b,\mu)]^*$, where $M(\omega_b,\mu)$ is given by (5.18), we first develop an expression for $M^m(\omega,\mu)$. If we define the polynomials $S_0(\mu), S_1(\mu), \ldots$ by the recursion formula

$$\begin{aligned} &S_k(\mu) = \omega\mu S_{k-1}(\mu) + (1-\omega)S_{k-2}(\mu), \qquad k \geq 2, \\ &S_0(\mu) = 1, \qquad S_1(\mu) = \omega\mu, \end{aligned} \tag{6.1}$$

then by a result of Young [7, p. 248] we have

$$M(\omega,\mu)^m = \begin{pmatrix} (1-\omega)S_{2m-2} & S_{2m-1} \\ (1-\omega)S_{2m-1} & S_{2m} \end{pmatrix}. \tag{6.2}$$

Notice that μ is purely imaginary. By (6.1) one can see that $S_{2k}(\mu)$ are real and S_{2k+1} purely imaginary. Also from the result of Young [7, p. 249, Eq. (4.7)] we have

$$S_k(\mu) = \sum_{j=0}^{k} \alpha_1^{k-j}\alpha_2^j = \begin{cases} \dfrac{\alpha_1^{k+1} - \alpha_2^{k+1}}{\alpha_1 - \alpha_2} & \text{if } \alpha_1 \neq \alpha_2, \\ (k+1)\alpha^k & \text{if } \alpha_1 = \alpha_2. \end{cases} \tag{6.3}$$

Here, α_1 and α_2 are the solution of the quadratic equation

$$\alpha^2 - \omega\mu\alpha + \omega - 1 = 0. \tag{6.4}$$

Now we prove that if $\omega = \omega_b$ and $r = (1 - \omega_b)$ then

$$S_k(i\bar{\mu}) = S_k(i\rho(B)) = (i)^k(r^{1/2})^k \cdot (k+1), \tag{6.5}$$

$$\max_{\substack{\mu = i\beta \\ -\bar{\mu} \leq \beta \leq \bar{\mu}}} |S_k(\mu)| = |S_k(i\bar{\mu})| = (k+1)(r^{1/2})^k. \tag{6.6}$$

Let $\mu = i\beta$; then the roots α_1 and α_2 of (6.4) are given by

$$\alpha_{1,2} = [i\beta\omega_b \pm \sqrt{-\beta^2\omega_b^2 - 4(\omega_b - 1)}]/2 = i[\beta\omega_b \pm \sqrt{\beta^2\omega_b^2 + 4(\omega_b - 1)}]/2.$$

By (1.9) we have

$$\bar{\mu}^2\omega_b^2 + 4(\omega_b - 1) = 0.$$

Thus, if $\beta = \bar{\mu}$, then $\alpha_1 = \alpha_2 = i\bar{\mu}\omega_b/2 = ir^{1/2}$. Hence (6.5) follows from (6.3). If $|\beta| \leq \bar{\mu}$, then $\beta^2\omega_b^2 + 4(\omega_b - 1) \leq 0$. Therefore, $|\alpha_1| = |\alpha_2| = (1 - \omega_b)^{1/2} = r^{1/2}$. Again by (6.3), (6.6) follows.

From (6.2) we have

$$\begin{aligned} &M^m(\omega,\mu)[M^m(\omega,\mu)]^* \\ &= \begin{pmatrix} (1-\omega)S_{2m-2} & S_{2m-1} \\ (1-\omega)S_{2m-1} & S_{2m} \end{pmatrix} \begin{pmatrix} (1-\omega)S_{2m-2} & -(1-\omega)S_{2m-1} \\ -S_{2m-1} & S_{2m} \end{pmatrix} \\ &= \begin{pmatrix} (1-\omega)^2 S_{2m-2}^2 - S_{2m-1}^2 & S_{2m}S_{2m-1} - (1-\omega)^2 S_{2m-1}S_{2m-2} \\ (1-\omega)^2 S_{2m-1}S_{2m-2} - S_{2m}S_{2m-1} & -(1-\omega)^2 S_{2m-1}^2 + S_{2m}^2 \end{pmatrix}. \end{aligned} \tag{6.7}$$

Evidently, the characteristic equation for $M^m(\omega_b,\mu)[M^m(\omega_b,\mu)]^*$ is

$$\lambda^2 - T_m(\omega_b,\mu)\lambda + \Delta = 0, \tag{6.8}$$

where

$$T_m(\omega_b,\mu) = (1-\omega_b)^2 S_{2m-2}^2 - S_{2m-1}^2 - (1-\omega_b)^2 S_{2m-1}^2 + S_{2m}^2 \tag{6.9}$$

and

$$\Delta = \det\{M^m(\omega_b,\mu)[M(\omega_b,\mu)]^*\} = r^{4m} = (1-\omega_b)^{4m} \tag{6.10}$$

by (5.18). Since $[T_m(\omega_b,\mu)]^2 - 4\Delta \geq 0$, because the eigenvalues of the Hermitian matrix $M^m(\omega_b,\mu)[M^m(\omega_b,\mu)]^*$ are real, it follows that for fixed Δ the modulus of the root of (6.8) is maximized when $|T_m(\omega_b,\mu)|$, considered as a function of μ, is maximized. But, by (6.5) and (6.6), $|T_m(\omega_b,\mu)|$ is maximized when $\mu = i\bar{\mu}$, and we have

$$\begin{aligned} |T_m(\omega_b, i\bar{\mu})| &= r^2 \cdot (2m-1)^2 r^{2m-2} + r^{2m-1} \cdot (2m)^2 \\ &\quad + r^2(2m)^2 r^{2m-1} + (2m+1)^2 r^{2m} \\ &= 2r^{2m}[1 + 2m^2(\sqrt{r} + r^{-1/2})^2]. \end{aligned} \tag{6.11}$$

Thus, from (6.8), (6.10), and (6.11) we have

$$(\lambda - r^{2m})^2 = 4m^2(r^{-1/2} + r^{1/2})^2 r^{2m} \cdot \lambda \tag{6.12}$$

and

$$\lambda - r^{2m} = 2m(r^{-1/2} + r^{1/2}) r^m \lambda^{1/2}. \tag{6.13}$$

Hence we have proved the following

THEOREM 6. *Under the assumptions of Theorem* 3, *we have*

$$\begin{aligned} \|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}} &= r^m\{m(r^{-1/2} + r^{1/2}) + [m^2(r^{-1/2} + r^{1/2})^2 + 1]^{1/2}\} \\ &= F_1(m), \end{aligned} \tag{6.14}$$

where

$$r = 1 - \omega_b, \qquad \omega_b = \frac{2}{1 + \sqrt{1 + \bar{\mu}^2}}, \qquad \bar{\mu} = \rho(B). \tag{6.15}$$

From (6.14) we know that for any $\bar{\mu} = \rho(B) > 0$

$$\operatorname*{limit}_{m \to \infty} \|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}} = \operatorname*{limit}_{m \to \infty} F_1(m) = 0.$$

But, for values of r close to unity, the function $F_1(m)$ increases initially before eventually decreasing. For r close to unity we have

$$\begin{aligned} \|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}} &\sim 2mr^m(r^{-1/2} + r^{1/2}) \\ &= 2mr^m(r^{-1}r^{1/2} + r^{-1}r \cdot r^{1/2}) \\ &\sim 4mr^{m-1}. \end{aligned} \tag{6.16}$$

On the other hand, we have

$$\|\mathscr{L}_{\omega_b}^m\| = \|M^m(\omega_b, i\bar{\mu})\| \sim mJ(M(\omega_b, i\bar{\mu}))r^{m-1}$$

by Theorem 3-7.1 [7, p. 85]. Here, $J(M(\omega_b, i\bar{\mu}))$ is the Jordan condition number of $M(\omega_b, i\bar{\mu})$. But by [7, Theorem 3-8.1, p. 89] we have

$$J(M(\omega_b, i\bar{\mu})) = \omega_b\bar{\mu} + (1 - \omega_b)_{\omega_b}\bar{\mu} = \omega_b\bar{\mu}(1 + 1 - \omega_b) = 2r^{1/2}(1 + r) \sim 4.$$

Hence,

$$\|\mathscr{L}_{\omega_b}^m\| \sim 4mr^{m-1}. \tag{6.17}$$

Therefore, $\|\mathscr{L}_{\omega_b}^m\|$ behaves like $\|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}}$.

Young [7, p. 255, Eq. (4.50)] has given m_0, the estimated number of iterations needed to reduce the $D^{1/2}$-norm of the error vector to a specified fraction ε of the $D^{1/2}$-norm of the initial error vector as follows:

$$\begin{aligned} m_0 &= \log((2\nu/\varepsilon) \cdot \log(2\nu/\varepsilon))/\log(1/r), \\ \nu &= \frac{r^{1/2} + r^{-1/2}}{\log(1/r)}. \end{aligned} \tag{6.18}$$

Final Remarks. (a) Since $\|\mathscr{L}_{\omega_b}\|_{D^{1/2}} \geq 1$ if $\bar{\mu} \geq 1$, one should expect that it may be better to use $\omega = \omega_+$ rather than $\omega = \omega_b$ in the initial steps. In this direction, an investigation is under way.

(b) By noting Theorem 6 and Theorem 7-4.1 of Young [7] one can find out that $\|\mathscr{L}_{\omega_b}^m\|_{D^{1/2}}$ for the nonsymmetric case and $\|\mathscr{L}_{\omega_b'}^m\|_{D^{1/2}}$ for the symmetric and positive definite case have the same expression in m and r. The only diference is that for the former,

$$(6.19) \qquad r = 1 - \omega_b = \frac{\rho^2(B)}{(1+\sqrt{1+\rho^2(B)})},$$

and for the latter,

$$(6.20) \qquad r = \omega_b' - 1 = \frac{\rho^2(B)}{(1+\sqrt{1-\rho^2(B)})}.$$

Especially for $m = 1$, we have

$$(6.21) \qquad \begin{aligned} \|\mathscr{L}_{\omega_b}\|_{D^{1/2}} = \|\mathscr{L}_{\omega_b'}\|_{D^{1/2}} &= r\{(r^{-1/2}+r^{1/2}) + [1+(r^{-1/2}+r^{1/2})^2]^{1/2}\} \\ &= r^{1/2}\{(1+r) + [r+(1+r)^2]^{1/2}\} = F(r). \end{aligned}$$

It is clear that $F(r)$ is an increasing function of r. In fact one can prove

LEMMA. *Let $F(r)$ be given by* (6.21). *Then*

$$F(r) < 1 \Leftrightarrow 0 \le r < r_0 = 1/(1+\sqrt{2})^2.$$

By means of the above lemma we can give another proof of Theorem 5. In fact, it follows from (6.19), (6.21) and the above lemma that $\|\mathscr{L}_{\omega_b}\|_{D^{1/2}} < 1$ if and only if $\rho^2(B)/(1+\sqrt{1+\rho^2(B)}) < 1/(1+\sqrt{2})^2$, or equivalently, $\rho(B) < 1$. Thus, Theorem 5 follows. However, we can give a similar result for the symmetric case. By noting (6.20), (6.21) and the above lemma, we have $\|\mathscr{L}_{\omega_b'}\|_{D^{1/2}} < 1$ if and only if $\rho^2(B)/(1+\sqrt{1-\rho^2(B)}) < 1/(1+\sqrt{2})^2$, or equivalently, $\rho(B) < 1/\sqrt{2}$. Thus we have proved

COROLLARY. *If A has the form* (1.7) *and is symmetric positive definite, then the $D^{1/2}$-norm of the corresponding optimum SOR iterative matrix $\mathscr{L}_{\omega_b'}$ is less than unity if and only if $\rho(B) < 1/\sqrt{2}$.*

To our knowledge, the result of the above corollary is new. However, it should be noted that the result can be deduced from Theorem 7-3.1 of Young [7].

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